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A NOTE ON THE THEOREM ON DIFFERENTIAL INEQUALITIES

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Abstract. It is proved that if a linear operator $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is nonpositive and for the initial value problem

$$u''(t) = \ell(u)(t) + q(t), \quad u(a) = c_1, \quad u'(a) = c_2$$

the theorem on *differential inequalities* is valid, then ℓ is an $a-$Volterra operator.

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The following notation is used throughout the paper.

$\mathbb{N}$ is the set of natural numbers.

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0, +\infty[$.

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u : [a,b] \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in [a,b]\}$.

$C([a,b];\mathbb{R}_+) = \{u \in C([a,b];\mathbb{R}) : u(t) \geq 0 \text{ for } t \in [a,b]\}$.

$\widetilde{C}([a,b];\mathbb{R})$ is the set of absolutely continuous functions $u : [a,b] \to \mathbb{R}$.

$\widetilde{C}'([a,b];\mathbb{R})$ is the set of functions $u \in \widetilde{C}([a,b];\mathbb{R})$ such that $u' \in \widetilde{C}([a,b];\mathbb{R})$.

$\widetilde{C}'_{loc}([a,b[\,;\mathbb{R})$ is the set of functions $u \in \widetilde{C}([a,b];\mathbb{R})$ such that $u' \in \widetilde{C}([a,\beta];\mathbb{R})$ for every $\beta \in \,]a,b[\,$.

$\widetilde{C}'_{loc}(]a,b[\,;\mathbb{R})$ is the set of functions $u \in \widetilde{C}([a,b];\mathbb{R})$ such that $u' \in \widetilde{C}([\alpha,\beta];\mathbb{R})$ for every $[\alpha,\beta] \subset \,]a,b[\,$.

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

$L([a,b];\mathbb{R}_+) = \{p \in L([a,b];\mathbb{R}) : p(t) \geq 0 \text{ for almost all } t \in [a,b]\}$.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

P_{ab} is the set of operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.

We will say that $\ell \in \mathcal{L}_{ab}$ is an $a-$Volterra operator if for arbitrary $b_0 \in]a,b]$ and $v \in C([a,b];\mathbb{R})$ satisfying the condition

$$v(t) = 0 \quad \text{for} \quad t \in [a, b_0]$$

we have

$$\ell(v)(t) = 0 \quad \text{for almost all} \quad t \in [a, b_0].$$

We will say that an operator $\Omega : L([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ is an $a-$Volterra operator, if for arbitrary $b_0 \in]a,b]$ and $q \in L([a,b];\mathbb{R})$ satisfying the condition

$$q(t) = 0 \quad \text{for almost all} \quad t \in [a, b_0]$$

we have

$$\Omega(q)(t) = 0 \quad \text{for} \quad t \in [a, b_0].$$

In what follows, the equalities and inequalities with integrable functions are understood to hold almost everywhere.

Consider the problem on the existence and uniqueness of a solution of the equation

$$u''(t) = \ell(u)(t) + q(t) \tag{1}$$

satisfying the initial conditions

$$u(a) = c_0, \quad u'(a) = c_1, \tag{2}$$

where $\ell \in \mathcal{L}_{ab}$, $q \in L([a,b];\mathbb{R})$ and $c_0, c_1 \in \mathbb{R}$. By a solution of the equation (1) we understand a function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfying this equation (almost everywhere) in $[a,b]$.

Along with the problem (1), (2) consider the corresponding homogeneous problem

$$u''(t) = \ell(u)(t), \tag{1_0}$$

$$u(a) = 0, \quad u'(a) = 0. \tag{2_0}$$

The following result is well-known from the general theory of boundary value problems for functional differential equations (see, e.g., [1, 2, 4, 5, 8]).

Theorem 1. *The problem* (1), (2) *is uniquely solvable iff the corresponding homogeneous problem* (1_0), (2_0) *has only the trivial solution.*

Definition 1. We will say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\widetilde{H}_{ab}(a)$ if for every function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfying

$$u''(t) \geq \ell(u)(t) \quad \text{for} \quad t \in [a,b], \tag{3}$$

and (2_0), the inequality

$$u(t) \geq 0 \quad \text{for} \quad t \in [a,b] \tag{4}$$

holds.

Remark 1. It follows from Definition 1 that if $\ell \in \widetilde{H}_{ab}(a)$, then the homogeneous problem (1_0), (2_0) has only the trivial solution. Therefore, according to Theorem 1 the problem (1), (2) is uniquely solvable. Moreover, the inclusion $\ell \in \widetilde{H}_{ab}(a)$ guarantees that if $q \in L([a,b];\mathbb{R}_+)$, then the unique solution of the problem (1), (2_0) is nonnegative.

Note also that $\ell \in \widetilde{H}_{ab}(a)$ iff a certain theorem on differential inequalities hold. More precisely, whenever $u, v \in \widetilde{C}'([a,b];\mathbb{R})$ satisfy the inequalities

$$u''(t) \leq \ell(u)(t) + q(t), \qquad v''(t) \geq \ell(v)(t) + q(t) \quad \text{for} \quad t \in [a,b],$$
$$u(a) = v(a), \qquad u'(a) = v'(a),$$

then

$$u(t) \leq v(t) \quad \text{for} \quad t \in [a,b].$$

In the paper [7], sufficient conditions are established guaranteeing the inclusion $\ell \in \widetilde{H}_{ab}(a)$. In particular, in [7, Theorem 1.3] the following proposition is proved.

Proposition 1. *Let* $-\ell \in P_{ab}$ *be an* $a-$*Volterra operator and let there exist a function* $\gamma \in \widetilde{C}'_{loc}(]a,b[\,;\mathbb{R})$ *satisfying*

$$\gamma''(t) \leq \ell(\gamma)(t) \quad \textit{for} \quad t \in [a,b], \tag{5}$$
$$\gamma(t) > 0 \quad \textit{for} \quad t \in]a,b[\,, \tag{6}$$
$$\gamma(a) + \lim_{t \to a+} \gamma'(t) \neq 0. \tag{7}$$

Then $\ell \in \widetilde{H}_{ab}(a)$.

Below we will prove (see Theorem 3) that in Proposition 1 the condition on ℓ to be $a-$Volterra operator is necessary. Analogous result for first order functional differential equations is proved in [3].

Before we formulate the main results, let us introduce the following definition.

Definition 2. Let the problem (1_0), (2_0) have only the trivial solution. Denote by Ω the operator, which assigns to every function $q \in L([a,b];\mathbb{R})$ the solution of the problem (1), (2_0).

Remark 2. From Theorem 1 it follows that the operator Ω is well defined. It is also clear that Ω is a linear operator which maps the set $L([a,b];\mathbb{R})$ into the set $C([a,b];\mathbb{R})$.

Remark 3. It follows from [5, Theorem 1.4.1] that the operator Ω is continuous (bounded) (see also [1, 4, 6]).

Remark 4. It immediately follows from Definitions 1 and 2 that if $\ell \in \widetilde{H}_{ab}(a)$, then the operator Ω is nonnegative, i.e., it transforms the set $L([a,b];\mathbb{R}_+)$ into the set $C([a,b];\mathbb{R}_+)$.

Theorem 2. *Let $-\ell \in P_{ab}$ and $\ell \in \widetilde{H}_{ab}(a)$. Then Ω is an $a-$Volterra operator.*

Proof. Let $t_0 \in \,]a,b[$ and let the function $q \in L([a,b];\mathbb{R})$ be such that

$$q(t) = 0 \quad \text{for} \quad t \in [a, t_0]. \tag{8}$$

We will show that

$$\Omega(q)(t) = 0 \quad \text{for} \quad t \in [a, t_0]. \tag{9}$$

Denote by u the solution of the problem (1), (2_0) and by v the solution of the problem

$$v''(t) = \ell(v)(t) + |q(t)|, \tag{10}$$
$$v(a) = 0, \quad v'(a) = 0. \tag{11}$$

According to Remark 1 (see also Remark 4) and the assumption $-\ell \in \widetilde{H}_{ab}(a)$, we have

$$v(t) \geq 0 \quad \text{for} \quad t \in [a,b], \tag{12}$$
$$u(t) \leq v(t) \quad \text{for} \quad t \in [a,b]. \tag{13}$$

Since $-\ell \in P_{ab}$, it follows from (10) and (12) that

$$v''(t) \leq |q(t)| \quad \text{for} \quad t \in [a, t_0]$$

Hence, on account of (8), (11) and (12), we obtain

$$v(t) = 0 \quad \text{for} \quad t \in [a, t_0]. \tag{14}$$

On the other hand, by virtue of (1), (10), (13), and the assumption $-\ell \in P_{ab}$, we get

$$(u(t) - v(t))'' = \ell(u - v)(t) + q(t) - |q(t)| \geq q(t) - |q(t)| \quad \text{for} \quad t \in [a, b].$$

Hence in view of (8) and (14) we get

$$u''(t) \geq v''(t) = 0 \quad \text{for} \quad t \in [a, t_0].$$

The latter inequality, together with (13), (14) and (2_0), implies

$$u(t) = 0 \quad \text{for} \quad t \in [a, t_0].$$

Consequently (since $u(t) = \Omega(q)(t)$ for $t \in [a, b]$), the equality (9) is fulfilled.
□

Theorem 3. *Let $-\ell \in P_{ab}$ and $\ell \in \widetilde{H}_{ab}(a)$. Then ℓ is an $a-$Volterra operator.*

Proof. Assume the contrary, let ℓ be not an $a-$Volterra operator. Then there exist $v_0 \in C([a, b]; \mathbb{R})$ and $b_0 \in]a, b[$ such that

$$v_0(t) = 0 \quad \text{for} \quad t \in [a, b_0]$$

and

$$\text{mes}\{t \in [a, b_0] : \ell(v_0)(t) \neq 0\} > 0.$$

Without loss of generality we can assume that

$$\text{mes}\{t \in [a, b_0] : \ell(v_0)(t) < 0\} > 0. \tag{15}$$

First we will show that

$$\Omega(\ell(|v_0|))(t) = 0 \quad \text{for} \quad t \in [a, b_0], \tag{16}$$

where Ω is the operator introduced in Definition 2.

Choose a sequence of functions $v_k \in \widetilde{C}'([a,b];\mathbb{R}),\ k \in \mathbb{N}$, such that

$$\lim_{k\to+\infty} \|v_k - |v_0|\|_C = 0 \tag{17}$$

and

$$v_k(t) = 0 \quad \text{for} \quad t \in [a, b_0], \quad k \in \mathbb{N}. \tag{18}$$

According to Remark 3 and (17), we get

$$\lim_{k\to+\infty} \|\Omega(\ell(v_k)) - \Omega(\ell(|v_0|))\|_C = 0. \tag{19}$$

It is clear that

$$v_k''(t) = \ell(v_k)(t) + q_k(t) \quad \text{for} \quad t \in [a,b], \quad k \in \mathbb{N}, \tag{20}$$

where

$$q_k(t) \stackrel{def}{=} v_k''(t) - \ell(v_k)(t) \quad \text{for} \quad t \in [a,b], \quad k \in \mathbb{N}. \tag{21}$$

Consequently,

$$v_k(t) = \Omega(q_k)(t) \quad \text{for} \quad t \in [a,b], \quad k \in \mathbb{N}. \tag{22}$$

It follows from (20)–(22) that

$$v_k(t) = \Omega(v_k'')(t) - \Omega(\ell(v_k))(t) \quad \text{for} \quad t \in [a,b], \quad k \in \mathbb{N}. \tag{23}$$

Hence, taking into account the fact that Ω is an $a-$Volterra operator (see Theorem 2) and the condition (18), we obtain

$$\Omega(\ell(v_k))(t) = -v_k(t) = 0 \quad \text{for} \quad t \in [a, b_0], \quad k \in \mathbb{N}.$$

Thus, in view of (19), we get the equality (16).

Let u be a solution of the problem (1), (2_0), where

$$q(t) = \begin{cases} -\ell(|v_0|)(t) & \text{for} \quad t \in\ [a, b_0[\\ 0 & \text{for} \quad t \in\ [b_0, b] \end{cases}. \tag{24}$$

It is evident that

$$u(t) = \Omega(q)(t) \quad \text{for} \quad t \in [a, b] \tag{25}$$

and

$$q(t) \geq 0 \quad \text{for} \quad t \in [a, b]. \tag{26}$$

Moreover, on account of the assumption $-\ell \in P_{ab}$, the inequality

$$\ell(|v_0|)(t) \leq \ell(v_0)(t) \quad \text{for} \quad t \in [a, b]$$

holds. Consequently, due to (15) and (24)

$$\text{mes}\{t \in [a, b_0] : q(t) > 0\} > 0. \tag{27}$$

According to Theorem 2, Ω is an $a-$Volterra operator. Hence by virtue of (16) and (24) we get from (25) that

$$u(t) = 0 \quad \text{for} \quad t \in [a, b_0]. \tag{28}$$

On the other hand, the inequality (26) and the assumption $\ell \in \widetilde{H}_{ab}(a)$ imply

$$u(t) \geq 0 \quad \text{for} \quad t \in [a, b]. \tag{29}$$

In view of (29) and the assumption $-\ell \in P_{ab}$, it follows from (1) that

$$u''(t) \leq q(t) \quad \text{for} \quad t \in [a, b].$$

Hence, on account of (24), we obtain

$$u''(t) \leq 0 \quad \text{for} \quad t \in [b_0, b].$$

The latter inequality, together with (28) and (29), yields

$$u(t) = 0 \quad \text{for} \quad t \in [a, b].$$

Therefore, it follows from (1) that $q \equiv 0$, which contradicts (27).

□

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